

Are quantization rules for horizon areas universal?

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Abstract

Doubts have been expressed on the universality of holographic/string-inspired quantization rules for the horizon areas of stationary black holes, or the products of their radii, already in simple 4-dimensional general relativity. Realistic black holes are not stationary but time-dependent. Using two examples of 4D general-relativistic spacetimes containing dynamical black holes for at least part of the time, it is shown that the quantization rules (even counting virtual horizons) cannot hold, except possibly at isolated instants of time, and do not seem to be universal.

1 Introduction

Recently, there has been some excitement in the research community working on the holographic principle and stringy/supergravity black holes following the observation that the products of Killing horizon areas for certain multi-horizon black holes are independent of the black hole mass and depend only on the quantized charges (supergravity and extra-dimensional black holes with angular momentum and electric and magnetic charges were considered) [1]. This literature, inspired by the holographic principle and string theories (although the results are not strictly derived from string theories), stems from the underlying idea that quantized products of areas depending on combinations of integers must carry the signature of some specific microphysics. This would not be too surprising if the area A of an horizon is related to its entropy S through the famous Bekenstein-Hawking formula $S = A/4$ (in units in which $c = \hbar = 1$) and corresponds to a statistical mechanics based on microscopic models counting microstates determined by quantum gravity. When there are outer (+) and inner (−) horizons, the quantization rules recurrent in the literature are

$$A_{\pm} = 8\pi l_{pl}^2 \left(\sqrt{N_1} - \sqrt{N_2} \right) \quad N_1, N_2 \in \mathbb{N}, \quad (1.1)$$

or

$$A_+ A_- = (8\pi l_{pl}^2)^2 N \quad N \in \mathbb{N}, \quad (1.2)$$

where l_{pl} is the Planck length [1]. $N_{1,2}$ are integers for supersymmetric extremal black holes but are related to the numbers of branes, antibranes, and strings in less simple situations [2]. A weaker rule states that the product of horizon areas is independent of the black hole mass and depends only on the quantized charges. These rules have somehow come to be considered universal for all types of black holes endowed with multiple horizons. A word of caution has been raised by Visser [3, 4] about the universality of formulas stating that products of areas are combinations of integers times a constant. Visser considered black holes in 4-dimensional general relativity, which should be the easiest to study and model from the statistical mechanics point of view, and found that in these situations products of areas do not give mass-independent quantities, nor are they related in a simple way to integers. Rather, it is quadratic combinations of the various horizon radii (with the dimensions of an area, which can be referred to as “generalized areas”) which generate mass-independent quantities and are, presumably, the best candidates to be quantized [3, 4]. Moreover, it is essential to include in these algebraic combinations also cosmological and virtual horizons in addition to the black hole horizons [4]. Virtual horizons are negative or imaginary roots of the equation locating the horizons (which, in non-asymptotically flat solutions of the Einstein equations, provides also cosmological horizons). The quantization rules break down also for general Myers-Perry black holes in dimension $D \geq 6$ and for Kerr-anti-de Sitter black holes with $D \geq 4$ [5].

In this note we point out a fact which induces even more caution in discussing the products of horizon areas. The horizons considered in the literature are Killing (and event) horizons.

Realistic black holes are not stationary if nothing else because, already at the semiclassical level, they emit Hawking radiation and the backreaction due to this effect changes their masses which become time-dependent, together with their horizon radii and areas. For astrophysical black holes the effect is completely negligible but it is not so for quantum black holes. Therefore, a timelike Killing vector will not be present and in realistic situations one should consider not Killing and event horizons, but other kinds of horizons. Dynamical horizons have received much attention in quantum gravity [6]; at present it seems that apparent horizons (“AH”s, see [7] for reviews) are the best and most versatile candidates for the notion of time-dependent “horizon” and it is claimed that thermodynamical laws can be associated with AHs [8]. In any case, AHs are used as proxies for event horizons in studies of gravitational collapse in numerical relativity [9]. AHs coincide with event horizons in stationary situations but, in dynamical situations, they are spacelike or even timelike instead. In the following we consider dynamical situations and we focus on AHs.

2 Toy models for dynamical black holes

Here we consider two toy models of *dynamical* black holes, which are implemented by setting them in a Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological background. In the first model, a McVittie spacetime, there are a black hole, a cosmological, and a virtual horizon. In the second model, the Sultana-Dyer solution of the Einstein equations, the “McVittie no-accretion condition” is relaxed to allow accretion of the cosmic fluid and then we have either two real horizons (a black hole and a cosmological horizon) or two virtual horizons. Our main point is that, in dynamical situations, even if combinations of AH radii which are mass-independent exist, they depend continuously on time and cannot be expressed as combinations of integers.

2.1 The McVittie spacetime

The McVittie metric [10] describes a black hole embedded in a FLRW universe, which is a truly dynamical spacetime.¹ Limiting ourselves, for simplicity, to a spatially flat FLRW background, the line element can thus be written in the form [11]

$$ds^2 = - \left[1 - \frac{2m}{R} - H^2(t) \right] dt^2 - \frac{2H(t)R}{\sqrt{1 - \frac{2m}{R}}} dt dR + R^2 d\Omega_{(2)}^2, \quad (2.1)$$

where m is a constant related to the mass of the central inhomogeneity, $d\Omega_{(2)}^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the metric on the unit 2-sphere, $H(t) \equiv \dot{a}(t)/a(t)$ is the Hubble parameter, $a(t)$ is the scale factor of the FLRW background, and an overdot denotes differentiation with respect to the

¹The special case of de Sitter space admits a timelike Killing vector and is locally static in the region between the black hole and the de Sitter cosmological horizons.

comoving time t . The locally static Schwarzschild-de Sitter-Kottler spacetime corresponds to $a(t) = \exp(\sqrt{\Lambda/3}t)$ and $H = \sqrt{\Lambda/3}$ (where $\Lambda > 0$ is the cosmological constant) and is a special case of the McVittie metric which can be obtained using a simple transformation of the time coordinate [12]. Assuming a perfect fluid stress energy tensor, the Einstein equations provide the energy density $\rho(t)$ and pressure $P(t, R)$ of the background fluid. Again for simplicity, let us restrict to a cosmic fluid which reduces to dust (equation of state parameter $w \equiv P/\rho = 0$) at spatial infinity, then

$$\rho(t) = \frac{3}{8\pi} H^2(t), \quad (2.2)$$

$$P(t, R) = \rho(t) \left(\frac{1}{\sqrt{1 - \frac{2m}{R}}} - 1 \right). \quad (2.3)$$

The inverse metric is

$$(g^{\mu\nu}) = \begin{pmatrix} -\frac{1}{1-2m/R} & -\frac{HR}{\sqrt{1-2m/R}} & 0 & 0 \\ -\frac{Hr}{\sqrt{1-2m/R}} & (1 - \frac{2m}{R} - H^2 R^2) & 0 & 0 \\ 0 & 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2 \sin^2 \theta} \end{pmatrix}. \quad (2.4)$$

For any spherically symmetric metric written in terms of the areal radius R , the AHs are located by solving the equation $\nabla^c R \nabla_c R = 0$ or $g^{RR} = 0$ [13]. In the special case of the Schwarzschild-de Sitter-Kottler spacetime, which is a special case of McVittie, this equation coincides with the horizon condition reported in [4] but, in the general case, the Hubble parameter is time-dependent instead of constant. This cubic equation

$$R^3 + \frac{R}{H^2(t)} - \frac{2m}{H^2(t)} = 0 \quad (2.5)$$

has three solutions which, under conditions specified below, correspond to a time-dependent black hole AH with (proper) radius $R_{BH}(t)$, a cosmological AH with radius $R_C(t)$, and a virtual AH with negative radius $R_V(t)$. The three roots are

$$R_{BH} = \frac{2H^{-1}}{\sqrt{3}} \sin \psi, \quad (2.6)$$

$$R_C = -R_V = H^{-1} \left(\cos \psi - \frac{1}{\sqrt{3}} \sin \psi \right), \quad (2.7)$$

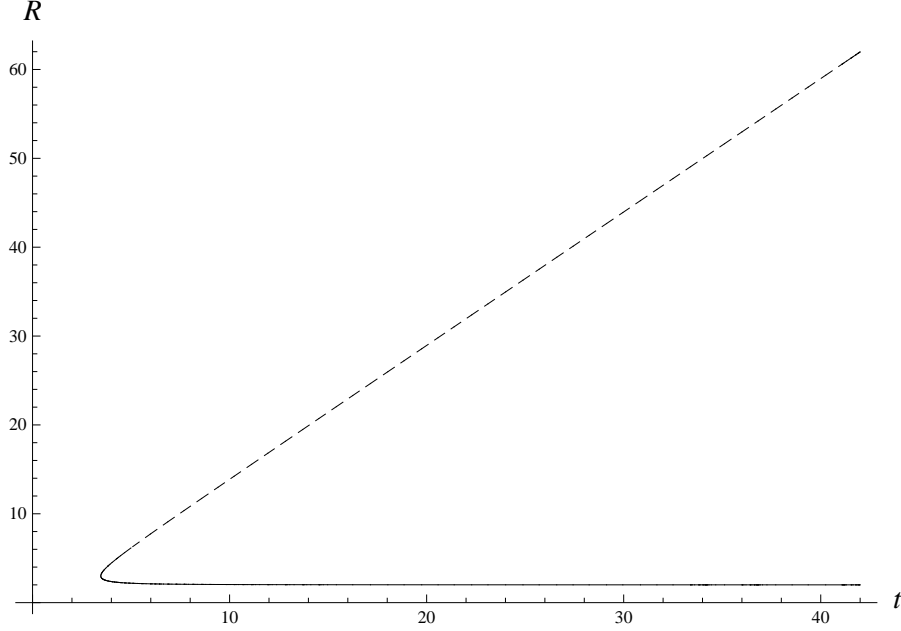


Figure 1: The proper radii of the AHs of a dust-dominated McVittie metric. At a critical time a cosmological (dashed curve) AH appears together with a black hole AH, the former expanding and the latter shrinking. There is also a virtual AH with negative radius (not represented in the figure).

with $\psi(t)$ given by $\sin(3\psi) = 3\sqrt{3}mH(t)$. Here m and H are both necessarily positive (we only consider expanding universes) and R_V is defined as the negative root. As discussed in [14], the condition for the black hole and cosmological AHs to exist is $0 < \sin(3\psi) < 1$, which corresponds to $mH(t) < 1/(3\sqrt{3})$ (and $mH(t) > 0$ which is always satisfied). Unlike the Schwarzschild-de Sitter-Kottler case where the Hubble parameter is a constant, this inequality will only be satisfied at certain times during the cosmological expansion and will be violated at other times. The threshold between these two regimes is the time at which $mH(t) = 1/(3\sqrt{3})$ (for a dust-dominated background with $H(t) = 2/(3t)$, this critical time is $t_* = 2\sqrt{3}m$). At early times $t < t_*$ it is $m > \frac{1}{3\sqrt{3}H(t)}$ and both $R_{BH}(t)$ and $R_C(t)$ are complex and therefore unphysical. In this case all the AHs are virtual. At the critical time $t = t_*$ it is $m = \frac{1}{3\sqrt{3}H(t)}$ and the AHs $R_{BH}(t_*)$ and $R_C(t_*)$ coincide at a real, physical location. There are then a single real AH at $\frac{1}{\sqrt{3}H(t)}$ and one virtual AH. At “late” times $t > t_*$ it is $m < \frac{1}{3\sqrt{3}H(t)}$, and both $R_{BH}(t)$ and $R_C(t)$ are real and, therefore, physical — there are two real and one virtual AHs. The dynamics of the black hole and cosmological AH radii as a function of comoving time is pictured in fig. 1.

The phenomenology of AHs appearing and disappearing in pairs appears to be rather general for black holes embedded in cosmological backgrounds, in both general relativity and alternative theories of gravity [15, 16, 17]. The physical reason why a pair of AHs suddenly appears in the McVittie spacetime (2.1) is discussed in [14]. The same phenomenology of fig. 1 is found for generalized McVittie metrics [18] and in Lemaître-Tolman-Bondi spacetimes ([19], see also [20]) describing black holes embedded in (spatially flat) FLRW universes.²

The Misner-Sharp-Hernandez mass M_{MSH} [22] of a sphere of areal radius R is defined for spherically symmetric spacetimes by [14]

$$M_{MSH} = m + \frac{4\pi G}{3} \rho R^3 \quad (2.8)$$

and coincides with the Hawking-Hayward quasi-local mass [23] in spherical symmetry. It is interpreted as the contribution of the black hole mass m (which is constant because of the “McVittie condition” $G_1^0 = 0$, which implies $T_1^0 = 0$ for the stress-energy tensor of the cosmic fluid and forbids accretion of the latter onto the black hole) and a contribution due to the energy of the cosmic fluid inside the sphere. Searching for generalized areas which are independent of the black hole mass, Visser’s discussion for the Schwarzschild-de Sitter-Kottler black hole can be repeated almost without changes. Including the virtual horizon in the count, it is straightforward to see that the quantities

$$R_V (R_{BH} + R_C) + R_{BH} R_C = -\frac{1}{H^2(t)} \quad (2.9)$$

and

$$(R_{BH} + R_C)^2 - R_{BH} R_C = \frac{1}{H^2(t)} \quad (2.10)$$

are independent of the black hole mass m . This situation can be regarded as a special case of Visser’s discussion [4] computing mass-independent combinations of AH radii whenever the Misner-Sharp-Hernandez mass is a Laurent polynomial of the areal radius R . This is clearly the case of the McVittie metric, see eq. (2.8). In the present case, the physical mass contained in a sphere is actually given by the Misner-Sharp-Hernandez notion, but the cosmic fluid here serves the only purpose of generating a cosmological background to make the central black hole dynamical and it seems that the relevant mass to consider when mass-independent quantities such as (2.9) and (2.10) are searched for is the black hole contribution m , not the total M_{MSH} . In any case, the AH radii identify different spheres and correspond to different Misner-Sharp-Hernandez masses $M_{MSH}^{(i)} = 2R_{AH}^{(i)}$ (from eq. (2.8)). Here we stick to m .

Following [4], we have included the virtual horizon to obtain the mass-independent quantity (2.9). Now, when the AH radii change with time, the combinations (2.9) and (2.10) are

²In the first case both decelerating and accelerating FLRW background universes are considered while in the second case, by necessity, only a dust-dominated background is considered.

not constant but depend on time: therefore, if they are expressed by combinations of integers at an initial time, they will not be combinations of integers immediately afterward. They could only be a combination of integers at times forming a set of zero measure in any time interval.

2.2 The Sultana-Dyer spacetime

The spherically symmetric line element of the Sultana-Dyer solution of the Einstein equations [21] can be cast in the form [24]

$$ds^2 = - \left(1 - \frac{2M}{R} - \frac{H^2 R^2}{1 - \frac{2M}{R}} \right) F^2 dT^2 + \frac{dR^2}{1 - \frac{2M}{R} - \frac{H^2 R^2}{1 - \frac{2M}{R}}} + R^2 d\Omega_{(2)}^2, \quad (2.11)$$

where R is the areal radius, $H \equiv \dot{a}/a$ is the Hubble parameter of the background FLRW universe with scale factor $a(t) = a_0 t^{2/3}$, and $F(T, R)$ is an integrating factor. The McVittie no-accretion condition of the cosmic fluid is removed and the spacetime is now sourced by two non-interacting perfect fluids (a null dust and a massive dust), as opposed to the single perfect fluid of the McVittie spacetime [21]. The two AHs of the Sultana-Dyer solution corresponding to an expanding FLRW background universe are [24] (see fig. 2)

$$R_{BH} = \frac{1 - \sqrt{1 - 8MH}}{2H}, \quad (2.12)$$

$$R_{CH} = \frac{1 + \sqrt{1 - 8MH}}{2H}, \quad (2.13)$$

where³ $M = ma(t)$ and m is a constant [24]. These AHs are both real when $8MH \leq 1$ and both virtual when $8MH > 1$ (in which case we label them R_{V1} and R_{V2}), which happens before a critical time t_* . $H \equiv \dot{a}/a$ is the Hubble parameter of the background FLRW universe.

The product of the horizon radii prescribed in [4] reduces to

$$R_{CH} R_{BH} = R_{V1} R_{V2} = \frac{2M}{H(t)} = 3ma_0 t^{5/3}, \quad (2.14)$$

which is time-dependent, in both cases in which the AHs are real or virtual. Following the same reasoning as in the previous section, we conclude that it cannot be expressed as a combination of integers. The Misner-Sharp-Hernandez/Hawking-Hayward mass of the black hole (when the latter exists) is [24]

$$M_{MSH} = \frac{R_{BH}}{2} = \frac{1 - \sqrt{1 - 8MH}}{4H}, \quad (2.15)$$

³Here the quantity M differs from the Misner-Sharp-Hernandez/Hawking-Hayward quasi-local mass [24].

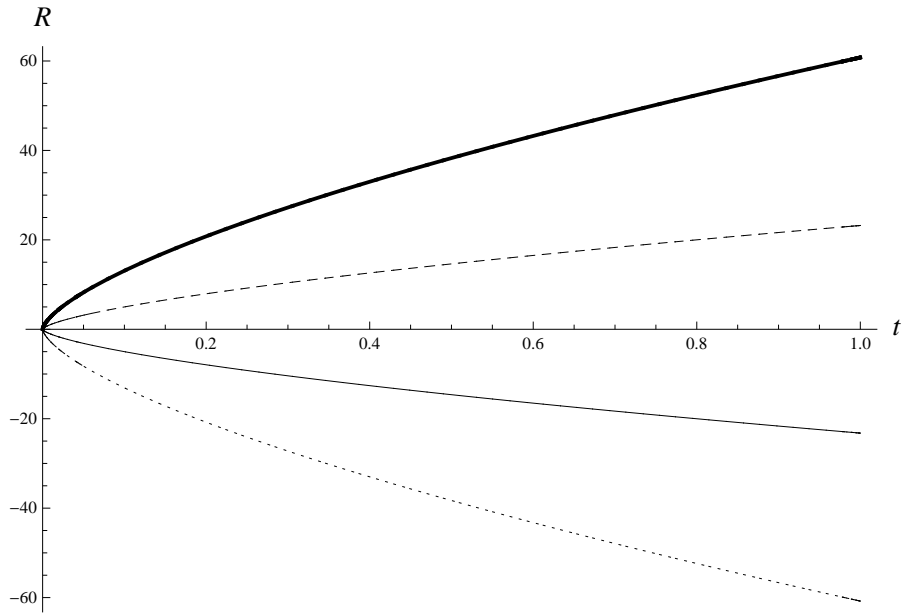


Figure 2: The radii of the AHs of the Sultana-Dyer spacetime (with t and R in units of m). There are always either two virtual AHs or two real AHs (a cosmological AH, thick curve, and a black hole AH, dashed curve, both appearing at a critical time).

it cannot be split in any simple way into a contribution due to the central inhomogeneity and one due to the cosmic fluid inside the sphere of radius R_{BH} , and it is time-dependent due to the radial energy flow onto the black hole. The product (2.14) can be rewritten as

$$R_{CH}R_{BH} = R_{V_1}R_{V_2} = 2M_{MSH} \left(\frac{1 + \sqrt{1 - 8MH}}{2H} \right) = 2M_{MSH}R_{CH} \quad (2.16)$$

and depends on the physical black hole mass.

3 Conclusions

The cosmological black holes reported here are just toy models for dynamical black hole horizons: the main point is that realistic black holes are time-dependent, not stationary. Therefore, far-reaching conclusions about the quantization of black hole horizon areas or of quantities which are quadratic in the radii of Killing horizons (generalized areas) may be misleading and may not correspond to realistic, time-dependent, situations. It is interesting to probe the conjecture about mass-independence and generalized area quantization using simple examples of time-varying black holes (without electric and magnetic charge) in 4-dimensional general relativity before approaching more complicated higher-dimensional black objects in supergravity. Exact solutions of the field equations of even simple Einstein theory describing time-varying black holes are not easy to find and we resort to the more well known cosmological black holes to provide examples of time-dependent black holes — the cosmological background is not essential here. In general, AHs depend on the spacetime foliation [25] but in the presence of spherical symmetry, to which we have restricted ourselves, this is not a problem.

For the McVittie metric (as well as for its special Schwarzschild-de Sitter-Kottler static case) there are generalized areas which are independent of the black hole mass. However, even if they can be expressed by $8\pi l_{pl}^2$ times a combination of integers at some initial time, this expression changes as time goes by. The corresponding quantity for the Sultana-Dyer black hole is mass- and time-dependent.

Variations on the theme can be contemplated. If only the black hole and the cosmological AHs are retained, and considered as physical, their area will be zero at all times $0 < t < t_*$; zero is an integer, all right, but this interpretation entails an entropy suddenly jumping from zero (describing a naked singularity in a FLRW background) to a value not reducible to a combination of integers and depending on the black hole mass. If the cosmological AH is excluded from the picture, then there remains only the black hole AH, the area of which is initially zero, then jumps to a positive value, and then decreases monotonically as time goes by (see fig. 1). More complicated black holes with multiple AHs will lend themselves to the consideration of more possible combinations of the AH radii, but probably the most sensible way to proceed is to include all AH radii, even virtual ones,⁴ when searching for quantizable, mass-independent

⁴The inclusion of virtual horizons in the count seems to attribute some physical meaning to them, but the

quantities, as done in [4]. When realistic time-dependent horizons are considered, however, the connection between products of areas and combinations of integers becomes even more speculative and perhaps it would be better to put it on a firmer ground or finding out its limits of validity before assuming it as a postulate or a necessary accessory of the holographic principle. This conclusion reinforces that of Visser [3, 4] that the black holes of 4-dimensional general relativity do not seem to reconcile with the usual quantization rules (1.1) and (1.2) and casts even more doubts on the universality of these expressions.

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latter it is not clear at present. It could be that the importance ascribed to virtual horizons is an artifact of the mathematics used to derive mass-independent generalized areas.

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